A New Quantized Input RLS, QI-RLS, Algorithm

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Abstract. Several modified RLS algorithms are studied in order to improve the rate of convergence, increase the tracking performance and reduce the computational cost of the regular RLS algorithm. . In this paper a new quantized input RLS, QI-RLS algorithm is introduced. The proposed algorithm is a modification of an existing method, namely, CRLS, and uses a new quantization function for clipping the input signal. We showed mathematically the convergence of the QI-RLS filter weights to the optimum Wiener filter weights. Also, we proved that the proposed algorithm has better tracking than the conventional RLS algorithm. We discuss the conditions which one have to consider so that he can get better performance of QI-RLS against the CRLS and standard RLS algorithms. The results of simulations confirm the presented analysis.

Keywords: Adaptive Filter, Recursive Least Square (RLS), Weiner Optimum Weights, Tracking.

1 Introduction

An adaptive filter is a filter which self-adjusts its transfer function according to an optimizing algorithm. Because of the complexity of the optimizing algorithms, most adaptive filters are digital filters that perform digital signal processing and adapt their performance based on the input signal. The Recursive Least Square (RLS) and the Least Mean Square (LMS) are two famous adaptive filtering algorithms [5]. They have attained its popularity due to a broad range of useful applications in such diverse areas as communications, radar, sonar, seismology, navigation and control systems, and biomedical electronics.

The optimization of convergence speed and tracking performance are open problems in adaptive filter theory. Fast convergence of the RLS has given rise to the development of the algorithms based on it [6,9,10,12].

The works reported in [1,2,3,7,8,14] have been done for increasing the real-time performance of the LMS algorithm using the sign of the input data and/or error during updating the filter weights. In the clipped RLS algorithm [11], the input signal is quantized into three levels of -1, 0, +1. They discussed the convergence and the computational complexity of their own CRLS algorithm.

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In the three levels clipping method the small domain signal is assumed as noise and it is obvious that it causes a lot of lost in input information in signals with low SNR[1] . In the proposed new clipped RLS, QI-RLS, algorithm the input has been clipped, such that we save more information and in the same time, the convergence and tracking performance of the proposed algorithm, are remained. The mathematical proofs and simulation results shows the better performance of the proposed method against Standard RLS and CRLS.

The variants of RLS are discussed in Section 2. The proposed new algorithm, which is a modification of the aforementioned algorithm, appears in Section 3. Section 4 deals with computer simulation issues. The final section presents conclusion and summarizes the main findings.

2 RLS Algorithm

In this section we review the Standard RLS and the CRLS Algorithm [11] which is the foundation of our proposed algorithm.

2.1 Standard RLS Algorithm

RLS algorithm has been explained in many literatures such as [5, 13]. In this section we review briefly this algorithm. The RLS predictor algorithm has been studied in [11] as:

$$W_{n+1} = W_n + R_n^{-1} X_n e_n \tag{1}$$

Where:

$$R_n^{-1} = \lambda R_{n-1}^{-1} - \frac{\lambda^{-2} R_{n-1}^{-1} X_n X_n^T (R_{n-1}^{-1})^T}{1 + \lambda^{-1} X_n^T R_{n-1}^{-1} X_n} \tag{2}$$

$$e_k = d_k - X_k^T W_k \tag{3}$$

$W_k = [w_k(1), w_k(2),..., w_k(N)]^T$ is the weight vector of the predictor, $X_k = [x_k(1), x_k(2),..., x_k(N)]$ is the vector of the input data sequence, which is assumed to be a stationary random process, N is the number of filter tapes, e_k is the estimation error and d_k is the desired response.

2.2 Clipped RLS Algorithm

Sadoghi et al. [11] have proposed Clipped-RLS as a variation of the standard RLS algorithm. They quantized the input signal with msgn(.) into a three-level signal. The msgn(.) function is described in equation 4 and is shown in Figure 1:

[1] Signal to Noise Ratio.

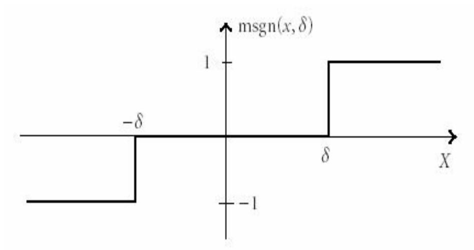

Fig. 1. Quantization scheme for the CRLS algorithm proposed in[11]

$$m\text{sgn}(x,\delta)=\begin{cases}1 & x>\delta\\ 0 & -\delta\le x\le\delta\\ -1 & x<\delta\end{cases} \tag{4}$$

According to msgn(.), the estimated input signal, $\hat{X}_n$ was replaced with X_n in Equations (2), (3). They also discussed the convergence and the computational complexity of their own CRLS algorithm.

In the next section we will explain the proposed new Quantized Input RLS (QI-RLS) and will discuss its performance, convergence and tracking.

3 The Proposed QI-RLS Algorithm

As we have seen in the previous section, the CRLS filter quantizes the input signal to three levels (Figure 1). Although it has been mentioned in [11] that it reduces the noise effect, but it is obvious that for inputs in range $[-\delta,+\delta]$, we have a lot of lost in input information. Because for that range, whole input assumed as noise. In the proposed new quantized input RLS, QI-RLS algorithm the input has been clipped such that we save all of the information for inputs in range $[-\delta,+\delta]$. Equation 4 shows our new clipping function that we named it $tgn(x,\delta)$ and is demonstrated in figure 2:

$$tgn(x,\delta)=\begin{cases}1 & x>\delta\\ x & -\delta\le x\le\delta\\ -1 & x<\delta\end{cases} \tag{5}$$

The adaptation equation can be written as:

$$W_{n+1}=W_n+R_n^{-1}X_n e_n \tag{6}$$

$$R_n^{-1}=\lambda R_{n-1}^{-1}-\frac{\lambda^{-2}R_{n-1}^{-1}\hat{X}_n\hat{X}_n^T(R_{n-1}^{-1})^T}{1+\lambda^{-1}\hat{X}_n^T R_{n-1}^{-1}\hat{X}_n} \tag{7}$$

$$e_k=d_k-X_k^T W_k \tag{8}$$

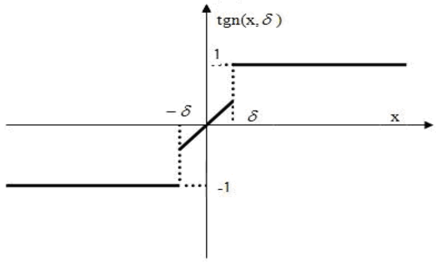

Fig. 2. Quantization scheme for the proposed clipping function, QI-RLS

Where $\hat{X}_n$ is the modified quantized input signal vector whose i^{th} component is: $\hat{x}_n(i) = tgn(x_n(i), \delta)$.

In the following subsections we explain the mathematical proof of the convergence and computational complexity reduction of the proposed method.

3.1 Convergence of QI-RLS

It is usual in adaptive filter literatures to prove the convergence of the filter weights to Weiner optimum weights. Theorem 1 proves the convergence of QI-RLS weights to Weiner weights.

Theorem 1: if the QI-RLS weights, W_n, is described by the recursive equation 6 and W_o is the Weiner optimum weight ,then W_n converges to W_o.

Proof:

We rewrite Equation (6) again: $W_{k+1} = W_k + R_k^{-1}\hat{X}_k e_k$

Where error is as follows:

$$e_k = d_k - X_k^T W$$

For convergence prove it is sufficient to show that: $\lim_{k\to\infty} E\{W_{k+1}\} = W_o$

Substituting e_k to Equation 6 yields: $W_{k+1} = W_k + R_k^{-1}(\hat{X}_k d_k - \hat{X}_k X_k^T W_k)$

Regarding the expectation of the above equation and Lemma 1 we have:

$$E\{W_{k+1}\} = E\{W_k\} + E\{R_k^{-1}\}(E\{\hat{X}_k X_k^T\}E\{W_k\})$$

$$= E\{W_k\} + E\{R_k^{-1}\}(\frac{\alpha'}{\sigma_x}E\{d_k X_k\} - \frac{\alpha'}{\sigma_x}E\{X_k^T X_k\}E\{W_k\})$$

$$\Rightarrow E\{W_{k+1}\} = E\{W_k\} + E\{R_k^{-1}\}\frac{\alpha'}{\sigma_x}(P - R.E\{W_k\}) \quad \textbf{(9)}$$

Where:

$$\alpha' = \sqrt{2/\pi}(1-\delta)\exp(-\delta^2/2\sigma_v^2) + \sigma_v erf(\delta/\sqrt{2}\,\sigma_v)$$

Here, we have to compute $E\{R_k^{-1}\}$.

By considering Lemma 2, $E\{R_k^{-1}\} = R^{-1}(1-\lambda)$, Equation (9) can be rewritten as follows:

$$E\{W_{k+1}\} = E\{W_k\} + R^{-1}(1-\lambda)\frac{\alpha'}{\sigma_x}(P - R.E\{W_k\})$$

$$\Rightarrow E\{W_{k+1}\} = E\{W_k\}(1-(1-\lambda)R^{-1}R\frac{\alpha'}{\sigma_x}) + (1-\lambda)R^{-1}.P.\frac{\alpha'}{\sigma_x}$$

The Weiner solution is $W_o = R^{-1}.P$. Hence:

$$E\{W_{k+1}\} = E\{W_k\}(1-(1-\lambda)\frac{\alpha'}{\sigma_x}) + (1-\lambda)\frac{\alpha'}{\sigma_x}W_o \tag{10}$$

Also, we know that:

$$\lim_{k\to\infty} E\{W_{k+1}\} = \lim_{k\to\infty} E\{W_k\}$$

By taking the limit of Equation (10) and after some simplification, we have:

$$\lim_{k\to\infty} E\{W_{k+1}\}\left(1-(1-(1-\lambda)\frac{\alpha'}{\sigma_x})\right) = (1-\lambda)\frac{\alpha'}{\sigma_x}W_o$$

$$\Rightarrow \lim_{k\to\infty} E\{W_{k+1}\} = W_o \ .$$

Hence, the proof is complete. □

3.2 Tracking Performance of QI-RLS

Tracking in filter theory, means tracking of the filter weights. According to [4,5,11], tracking is a steady-state phenomenon that is different from the convergence, which is a transient phenomenon. In general, convergence and tracking are two different properties. That is, if an algorithm has good convergence, its tracking ability is not necessarily fast and vice versa. In the tracking phase, a reasonable assumption is that the optimum weights vary according to a first-order Markov process [5], and the filter must track these weights. The following relation shows the variation of the filter's optimum weights:

$$\begin{aligned} W_{n+1}^{*} &= aW_n^{*} + \omega_n \\ d_n &= W_n^{*T} X_n + \upsilon_n \end{aligned} \tag{11}$$

Where a is a constant and ω_n is the process noise vector in the nth step, which has zero mean, and υ_n is the measurement noise, which is assumed to be white Gaussian with zero mean and variance σ_v^2 .

Missadjustment criterion as shown in Equation (12) is a usable measure for tacking performance [5]:

$$M = \frac{E\{\left|\omega_n^T X_n\right|^2\}}{E\{\left|\upsilon_n\right|^2\}} \tag{12}$$

In the following theorem we will show the relation between our QI-RLS and the conventional RLS according to this criterion.

Theorem 2: Let M_{QI-RLS} and M_{RLS} are QI-RLS and conventional RLS miss adjustment criterion, respectively, then:

$$M_{QI-RLS} = \left(\frac{\alpha'}{\sigma_x}\right)^2 M_{RLS} \quad \textbf{(13)}$$

Where:

$$\alpha' = \sqrt{2/\pi}(1-\delta)\exp(-\delta^2/2\sigma_x^2) + \sigma_x erf(\delta/\sqrt{2}\,\sigma_x) .$$

Proof:

$$E\{|\omega_n^T \hat{X}_n|^2\} = E\{\omega_n^T \hat{X}_n \hat{X}_n^T \omega_n\}$$

With suppose the independency between ω_n and $\hat{X}_n$. By using Lemma 1 we have:

$$E\{|\omega_n^T \hat{X}_n|^2\} = E\{\omega_n^T \hat{X}_n \hat{X}_n^T \omega_n\} = E\{\omega_n^T \hat{X}_n\} E\{\hat{X}_n^T \omega_n\}$$

$$= \left(\frac{\alpha'}{\sigma_x} E\{\omega_n^T X_n\}\right)\left(\frac{\alpha'}{\sigma_x} E\{X_n^T \omega_n\}\right) = \left(\frac{\alpha'}{\sigma_x}\right)^2 E\{|\omega_n^T X_n|^2\}$$

By dividing the equation to $E\{|\upsilon_n|^2\}$:

$$\frac{E\{|\omega_n^T \tilde{X}_n|^2\}}{E\{|\upsilon_n|^2\}} = \left(\frac{\alpha'}{\sigma_x}\right)^2 \frac{E\{|\omega_n^T X_n|^2\}}{E\{|\upsilon_n|^2\}} \quad \Rightarrow M_{QI-RLS} = \left(\frac{\alpha'}{\sigma_x}\right)^2 M_{RLS} . \ \square$$

In an instance filter, the reduction of the missadjustment measure means the increase of tracking performance [5]. Hence, from theorem 2 it is obvious that if $\alpha' < \sigma_x$, tracking performance of QI-RLS is better than RLS. But it does not discuss conditions which one have to consider so that he can get better performance of QI-RLS. Theorem 3 describes the suitable parameter values.

Theorem 3: if $\delta = \sqrt{2}\sigma_x$ then Tracking performance of QI-RLS algorithm is better than conventional RLS for $\sigma_x > 0.513$. Where σ_x is the variance of input signal, and δ is the parameter of $tgn(x,\delta)$.

Proof: It is obvious that, if the missadjustment of QI-RLS is lower than the missadjustment of conventional RLS, then tracking performance of QI-RLS is better than RLS. Regarding to results of theorem 2, we have:

$$M_{NC-RLS} = \left(\frac{\alpha'}{\sigma_x}\right)^2 M_{RLS}$$

Where $\alpha' = \sqrt{2/\pi}(1-\delta)\exp(-\delta^2/2\sigma_v^2) + \sigma_v erf(\delta/\sqrt{2}\,\sigma_v)$

Now, it is sufficient to show that if $\delta = \sqrt{2}\sigma_x$ then $\alpha' < \sigma_x$, hence:

$$\alpha' = \sqrt{2/\pi}(1-\sqrt{2}\sigma_x)\exp(-(\sqrt{2}\sigma_x)^2/2\sigma_x^2) + \sigma_x erf(\sqrt{2}\sigma_x/\sqrt{2}\,\sigma_x) < \sigma_x$$

After some simplification we have:

$$\sigma_x > \frac{\sqrt{2/\pi}\,\exp(-1)}{1 - erf(1) + 2/\sqrt{\pi}\,\exp(-1)} \approx 0.513$$

So, considering $\sigma_x > 0.513$ and $\delta = \sqrt{2}\sigma_x$ implies $\alpha' < \sigma_x$, and the proof is complete. □

Following section shows some experimental results demonstrating the better performance of the QI-RLS against to some others.

4 Experimental Results: Predicting a noisy chirp signal

As mentioned earlier on theorem 1, the proposed filter weights converges to Wiener optimum weights. The Wiener optimum weights are computed as follow:

$$W_o = R^{-1}P \tag{14}$$

Where: $R = E\{X_k X_k^T\}$ and $P = E\{d_k X_k\}$.

Figure 3 shows the weight tracking of our method, standard RLS and CRLS [5] for tracking a noisy chirp sinusoid signal, which was shown in Figure 4. As Equation (14) shows, for computing the Wiener Weights, all of data must be in hand. For comparison purpose, at every time, we used the incoming data until that time. The following pseudo code shows our used algorithm for computing Wiener Optimum weights[1].

```
X: Input Signal
Y: Desired Output where Ly=Length(Y)
N: Number of last input for prediction
For k= N: Ly
      x_k = X_(k-N+1),…,X_(k);
      P = P + Y_(k)*x_k;
      R = R + x_k*x_k^T;
      Weiner(k) =R^-1 * P;
End
OptWeights = R^-1 * P;   //Optimum Weiner Weights
```

[1] The original code is about to Rich Kozick.

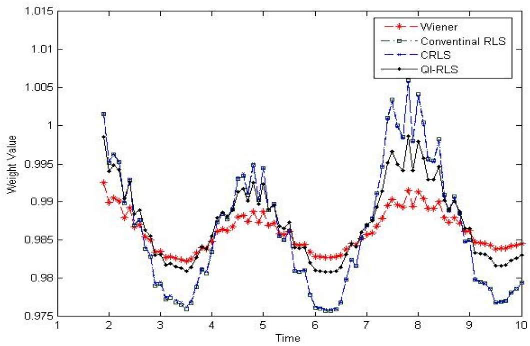

Fig. 3. Weight Tracking Comparison between Wiener filter, RLS, CRLS and the proposed QI-RLS.

As can be seen in Figure 3 the weights of the QI-RLS are closer to Wiener weights comparing to others.

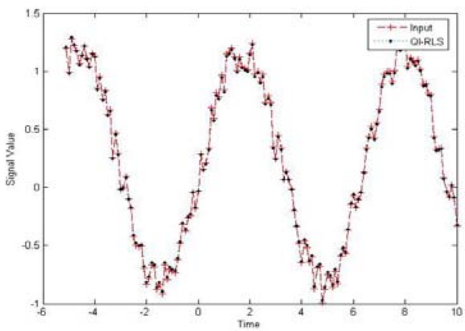

Fig. 4. Signal Tracking: Tracking a noisy chirp sinusoid signal by QI-RLS.

Also we saw in Theorem 3, that the tracking performance of QI-RLS algorithm is better than conventional RLS, based on some conditions. The result of running our algorithm for 40 noisy signals, regarding the theorem 3 conditions, shows that the proposed method can produce better results comparing to some others.

The mean square error of an estimator $\widehat{\theta}$ of a parameter θ represented by (15) is a widely used criterion for comparing estimator algorithms.

$$MSE(\widehat{\theta}) = E[(\widehat{\theta} - \theta)^2] \quad \textbf{(15)}$$

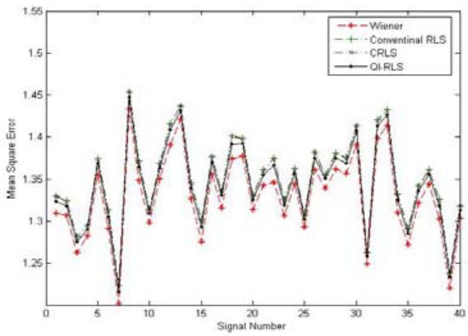

Fig. 5. Comparing MSE of the proposed QI-RLS, conventional RLS and CRLS predicted signals with the original noisy signal.

Figure 5 shows the mean squared error of the proposed QI-RLS, conventional RLS and CRLS predicted signals with the original noisy signal. The original signal is considered as θ and the predicted signal as $\widehat{\theta}$ in Equation (15).The algorithms were run on 40 random chirp sinusoidal signals and for every signal the error estimation are computed. As can be seen the related MSE of the proposed filter and the CRLS filter are lower than conventional RLS. Closing the result of the proposed method and the CRLS, in Fig 5, makes it hard to distinguish which of them is really better. Hence we have used another criterion in Figure 6 for better demonstration.

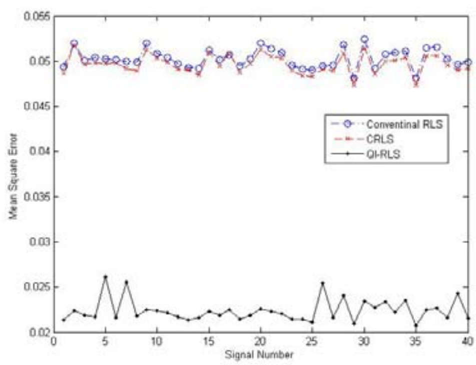

Fig. 6. Weight estimation error, MSE of the difference weight vector between proposed QI-RLS, conventional RLS, CRLS weights and Wiener optimum weights.

5 Conclusion

Prediction is a major part of tracking algorithms. One of the most commonly used algorithms in prediction is the RLS algorithm. In this paper we proposed a new variant of the RLS, namely, the QI-RLS algorithm. This algorithm uses a quantization scheme which involves the threshold clipping of the input signal. Mathematical analysis shows the convergence of the filter weights to the optimum Wiener weights. Also we showed under which circumstances tracking performance of QI-RLS algorithm is better than conventional RLS. Experimental results on predicting a noisy chirp signal demonstrated the good performance of the proposed algorithm.

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Appendix

Lemma 1. If two random variables u and v both have a Gaussian distribution $N(0,\sigma_u)$ and $N(0,\sigma_v)$ respectively and $E\{uv\}=\rho\sigma_u\sigma_v$, $\hat{v}=tgn(v,\sigma)$ then:

$$E\{u\hat{v}\}=\frac{\alpha'}{\sigma_v}E\{uv\}$$

Where $\alpha'=\sqrt{2/\pi}(1-\delta)\exp(-\delta^2/2\sigma_v^2)+\sigma_v erf(\delta/\sqrt{2}\,\sigma_v)$

Proof:

We define the random variable $z=\frac{u}{\sigma_u}+\rho\frac{v}{\sigma_v}$

Now we have:

$$E\{zv\}=E\left\{\left(\frac{u}{\sigma_u}-\rho\frac{v}{\sigma_v}\right)v\right\}=E\left\{\frac{uv}{\sigma_u}\right\}-E\left\{\frac{\rho v^2}{\sigma_v}\right\}$$

With regard to assumption of the theorem $E\{zv\}=\frac{\rho\sigma_u\sigma_v}{\sigma_u}-\frac{\rho\sigma_v^2}{\sigma_v}=0$

Therefore z and v are uncorrelated. Here we show that z and $\hat{v}$ are uncorrelated too. We have to show $E\{z\hat{v}\}=0$. At first we prove that $E\{\hat{v}\}=0$.

$$E\{\hat{v}\}=\int_{-\infty}^{\infty}\hat{v}f(\hat{v})d\hat{v}=\int_{-\infty}^{\infty}\hat{v}\exp(\frac{-\hat{v}^2}{\sigma_{\hat{v}}^2})d\hat{v}=\int_{-\infty}^{-\delta}-\exp(\frac{-1}{\sigma_{\hat{v}}^2})d\hat{v}+\int_{-\delta}^{\delta}v\exp(\frac{-v^2}{\sigma_{\hat{v}}^2})dv+\int_{\delta}^{\infty}\exp(\frac{-1}{\sigma_{\hat{v}}^2})d\hat{v}=0$$

$$E\{z\hat{v}\}=E\{z\}E\{\hat{v}\}=0\Rightarrow E\left\{\left(\frac{u}{\sigma_u}-\rho\frac{v}{\sigma_v}\right)\hat{v}\right\}=0\Rightarrow\frac{1}{\sigma_u}E\{u\hat{v}\}=\frac{\rho}{\sigma_v}E\{v\hat{v}\}$$

Therefore, we have:

$$E\{u\hat{v}\}=\rho\frac{\sigma_u}{\sigma_v}E\{v\hat{v}\} \tag{A-1}$$

On the other hand

$$v\hat{v}=v\times tgn(v,\delta)=\begin{cases}|\mathrm{v}|, & |\mathrm{v}|>\delta\\ v^2, & |v|\le\delta\end{cases}$$

$$E\{v\hat{v}\}=\int_{-\infty}^{\infty}v\hat{v}\frac{1}{\sqrt{2\pi}\delta_v}\exp(\frac{-v^2}{\sigma_v^2})dv$$

$$=\int_{-\infty}^{-\delta}|v|\frac{1}{\sqrt{2\pi}\delta_v}\exp(\frac{-v^2}{2\sigma_v^2})dv+\int_{-\delta}^{\delta}v^2\frac{1}{\sqrt{2\pi}\delta_v}\exp(\frac{-v^2}{2\sigma_{\hat{v}}^2})dv+\int_{\delta}^{\infty}|v|\frac{1}{\sqrt{2\pi}\delta_v}\exp(\frac{-v^2}{2\sigma_v^2})dv$$

$$\Rightarrow E\{v\hat{v}\}=\frac{2}{\sqrt{2\pi}\delta_v}\int_{+\delta}^{+\infty}v\exp(\frac{-v^2}{2\sigma_v^2})dv+\frac{1}{\sqrt{2\pi}\delta_v}\int_{-\delta}^{+\delta}v^2\exp(\frac{-v^2}{2\sigma_{\hat{v}}^2})dv$$

Here we calculate the above two terms:

$$\frac{2}{\sqrt{2\pi}\delta_v}\int_{+\delta}^{+\infty}v\exp(\frac{-v^2}{2\sigma_v^2})dv=\sqrt{\frac{2}{\pi}}\sigma_v\exp(\frac{-\delta^2}{2\sigma_v^2})$$

$$\frac{1}{\sqrt{2\pi}\delta_v}\int_{-\delta}^{+\delta} v^2 \exp(\frac{-v^2}{2\sigma_{\hat{v}}^2})dv = \frac{1}{\sqrt{2\pi}\delta_v}\{-2\delta \exp(\frac{-\delta^2}{2\sigma_{\hat{v}}^2})\sigma_v^2 + \sigma_v^3 erf(\frac{\delta}{\sqrt{2}\sigma_v})\}$$

$$= -\delta\sqrt{\frac{2}{\pi}}\sigma_v \exp(-\frac{\delta^2}{2\sigma_v^2}) + \sigma_v^2 erf(\frac{\delta}{\sqrt{2}\sigma_v})$$

Substituting the 2,3 in 1, and after some simplifications we have:

$$E\{v\hat{v}\} = \sqrt{\frac{2}{\pi}}\sigma_v(1-\delta)\exp(\frac{-\delta^2}{2\sigma_v^2}) + \sigma_v^2 erf(\frac{\delta}{\sqrt{2}\sigma_v}) \quad \text{(A-2)}$$

Now we can compute $E\{u\hat{v}\}$.Regarding (A-1) and (A-2) we have:

$$E\{u\hat{v}\} = \frac{\rho\sigma_u}{\sigma_v}\left\{\sqrt{\frac{2}{\pi}}\sigma_v(1-\delta)\exp(\frac{-\delta^2}{2\sigma_v^2}) + \sigma_v^2 erf(\frac{\delta}{\sqrt{2}\sigma_v})\right\}$$

$$E\{u\hat{v}\} = \rho\sigma_u\sigma_v\left\{\frac{1}{\sigma_v}\sqrt{\frac{2}{\pi}}(1-\delta)\exp(\frac{-\delta^2}{2\sigma_v^2}) + erf(\frac{\delta}{\sqrt{2}\sigma_v})\right\} \Rightarrow$$

$$E\{u\hat{v}\} = \frac{1}{\sigma_v}E\{uv\}\left\{\sqrt{\frac{2}{\pi}}(1-\delta)\exp(\frac{-\delta^2}{2\sigma_v^2}) + \sigma_v erf(\frac{\delta}{\sqrt{2}\sigma_v})\right\}$$

Hence with defining α' as follows: $\alpha' = \sqrt{2/\pi}(1-\delta)\exp(-\delta^2/2\sigma_v^2) + \sigma_v erf(\delta/\sqrt{2}\,\sigma_v)$

So: $E\{u\hat{v}\} = \frac{\alpha'}{\sigma_v}E\{uv\}$. □

Lemma 2: if $R_k = \sum_{i=1}^{k}\lambda^{k-i}X_iX_i^T$ and $R = E\{X_iX_i^T\}$ then $E\{R_k^{-1}\} = R^{-1}(1-\lambda)$.

Proof:

The expectation of R_k will be: $E\{R_k\} = R_k = \sum_{i=1}^{k}\lambda^{k-i}E\{X_iX_i^T\}$

According to Eleftheriou and Falconer's theorem [15], $E\{R_k\} = R_k$ and, regarding to lemma assumption, $R = E\{X_iX_i^T\}$, we have:

$$E\{R_k\} = R_k = \sum_{i=1}^{k}\lambda^{k-i}R = (1+\lambda+\lambda^2+...+\lambda^{k-1})R$$

If $k \to \infty$ and $\lambda < 1$ then: $E(R_k) = R_k = 1/(1-\lambda) R$

Taking the inverse of the above equation leads to: $E(R_k^{-1}) = R^{-1}(1-\lambda)$

and the proof is complete. □